Numerical methods in cosmology III

Correlation functions and fractals

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N-body problems

Galaxy clusters N = 10^2 - 10^3

Correlations

Star clusters 10^5 – 10^6
Galaxies up to 10^13

Statistics

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N-point correlation function

N point-like objects in 3-space V

Probability of finding one object in volume element dV at given position

$$dP = n dV$$

n is concentration averaged over volume after N events.

Probability of finding two points in a pair of volume elements dV_1 and dV_2

$$dP = n^2 [1 + \xi(r/r_0)]\, dV_1 dV_2$$

$\xi(r)$ is two-point spatial correlation function.

r_0 is the characteristic clustering length

$$\xi = (r_0/r)^{\gamma}$$

We consider normalized $r \rightarrow r/r_0$

For a number of points in some average ball B is

$$< N >= n^2[\int_B (1 + \xi(r))dV]$$

For Poisson process $\xi(r) = 0$

positive correlation $\xi > 0$

anti-correlation $-1 \leq \xi < 0$

Connection between the spatial and angular correlation functions: Limber equation

The two-point angular correlation function is

$$dp = \mathcal{N}^2 d\Omega_1 d\Omega_2 [1 + \omega(\vartheta_{12})]$$

where $\omega(\theta_{12})$ is the angular correlation function, $d\Omega_1$, $d\Omega_2$ are the celestial angle elements at angular distance θ_{12}; N is mean surface number density of objects.

Probability of finding two objects in two volume elements dV_1 and dV_2

$$dP(dV_1, dV_2) = n^2[1 + \xi(r_{12})]r_1^2 r_2^2 d\Omega_1 d\Omega_2 dr_1 dr_2$$

$$dV = r^2 dr d\Omega$$

$$dp = \mathcal{N} d\Omega$$

$$\mathcal{N} = n \int r^2 dr$$

$$dp = \int \frac{dP}{dr} dr = n d\Omega \int r^2 dr$$

Angular correlation function

$$\omega(\vartheta) = \frac{\int \xi(r_{12}) r_1^2 r_2^2 dr_1 dr_2}{[\int r^2 dr]^2}$$

New radial variables *u* and *y*

$$u = (r_1 - r_2)/2 \, , \qquad y = (r_1 + r_2)/2$$
$$r_1 = y + u/2 \, , \qquad r_2 = y - u/2 \, ,$$

$$\omega(\vartheta) = \frac{\int (y^2 + \frac{u^2}{4})^2 \xi(\sqrt{u^2 + 2(y^2 + \frac{u^2}{4})(1 - \cos\vartheta)}) dy du}{[\int y^2 dy]^2}$$

In limit when the relative distance between two objects is much smaller than their distances from observer

$$| u | \ll y, \ (\Leftrightarrow | r_1 - r_2 | \ll (r_1 + r_2)/2)$$

$$\vartheta \ll 1 \ \rightarrow \ 2(1 - \cos\vartheta) \approx \vartheta^2$$

Then

$$\omega(\vartheta) = \frac{\int y^4 \xi(\sqrt{u^2 + y^2\vartheta^2}) dy du}{[\int y^2 dy]^2}$$

Assign a statistical weight $\varphi(r)$ to the radial coordinate r

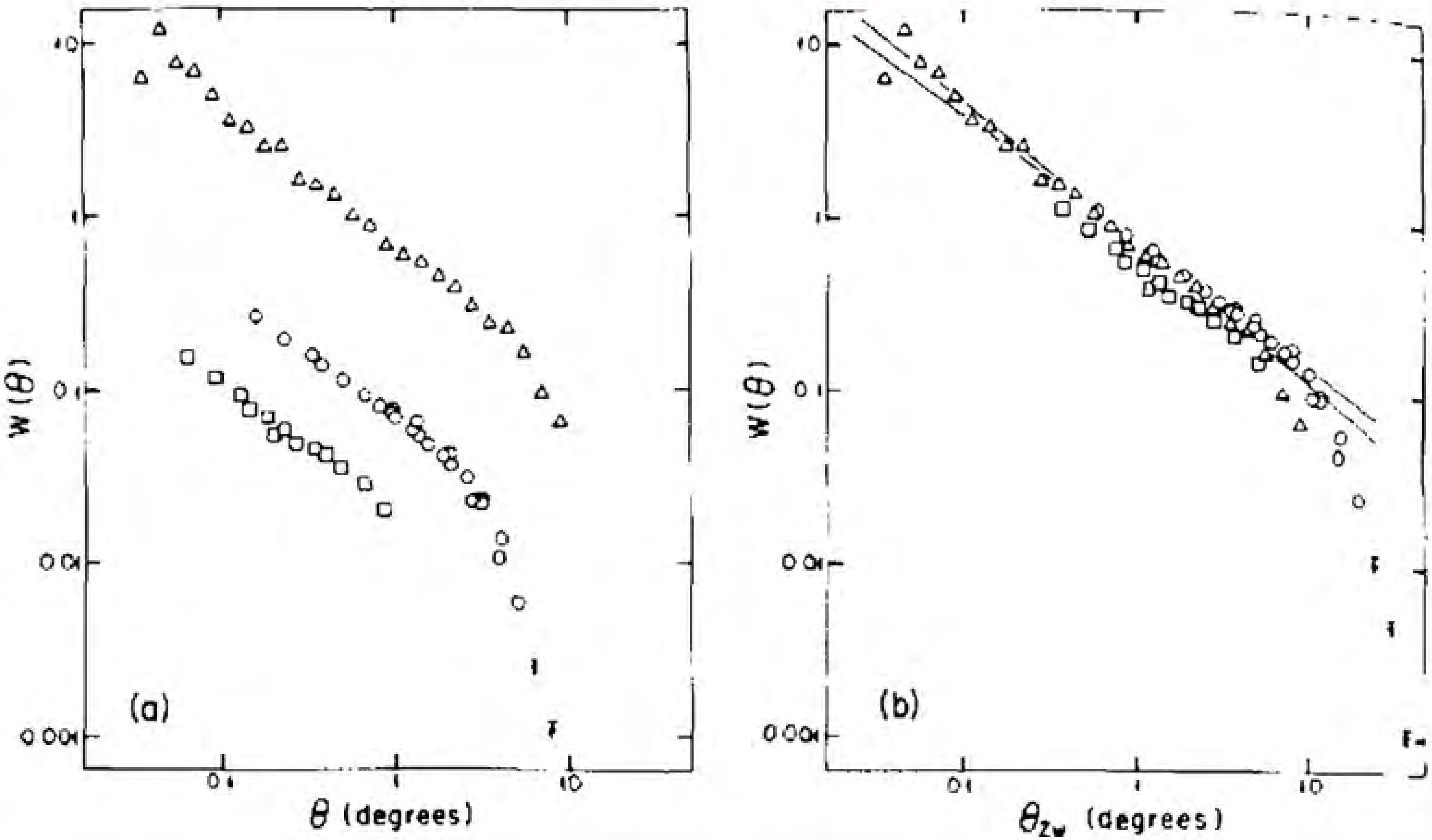
$$dr_i \to \varphi(r_i)dr_i$$

Relation between spatial and angular two-point correlation functions: Limber equation

$$\omega(\vartheta) = \frac{\int r_1^2 r_2^2 \varphi(r_1)\varphi(r_2)\xi(r_{12})dr_1 dr_2}{\left[\int \varphi(r) r^2 dr\right]^2}$$

or for *u* and *y*

$$\omega(\vartheta) = \frac{\int y^4 \varphi(y)\xi(\sqrt{u^2 + y^2\vartheta^2})dy\,du}{\left[\int \varphi^2(y) y^2 dy\right]^2}$$



Galaxy clusters: angular two-point correlation function. Panel (b) is the result of applying homogeneous scaling law to the correlation function estimates in panel (a) from catalogs at three different depths.

Fractals

- A set with the property of self-similarity, that is uniformity in the different measurement scales. Any part of a fractal repeats the whole set as a whole.

- Fractals have fractional dimension. They are distinguished from regular geometric figures which have integer dimension.

- Term "fractal" was first used by mathematician Benoît Mandelbrot in 1975.

- Fractal Zoom Mandelbrot Corner https://www.youtube.com/watch?v=G_GBwuYuOOs

Hausdorff dimension

For a set *A* embedded in the Euclidean space R^n the family of all countable coverings of formed by sets with diameter less or equal than a given positive number $r > 0$ is

$$\Upsilon_A^r = \{\{B_i\}_{i\in I} \mid A \subseteq \bigcup_{i\in I} B_i \mid r_i \le r\}$$

B_i being the sets belonging to each r-covering of A, with diameter rio The β – Hausdorff dimension of A is defined as

$$H^\beta(A) = \lim_{r\to 0} \inf_{\Upsilon_A^r} \sum_i r_i^\beta$$

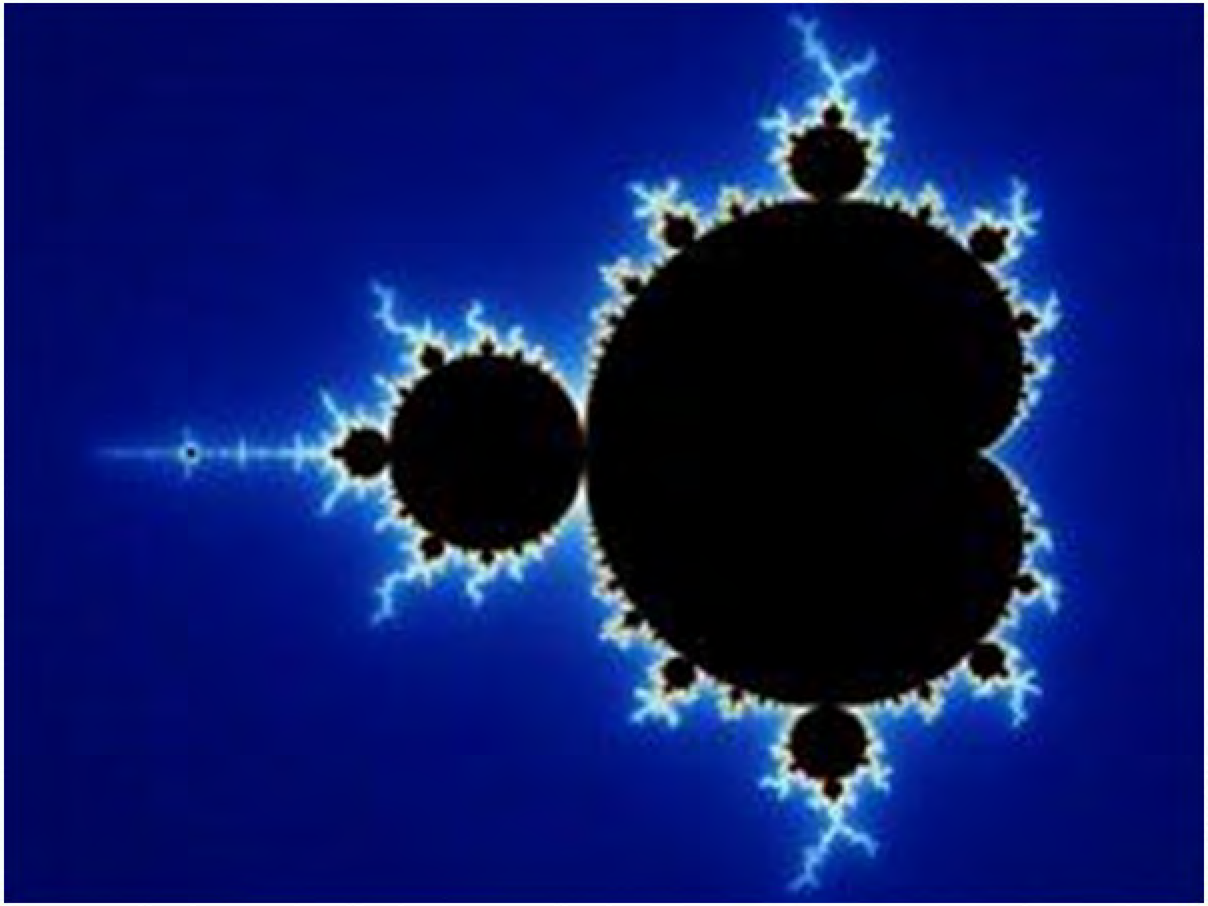
Considering the moments of the diameters of the covering sets, take the *infimum* among all possible coverings and find the limit when the upper bound of the diameters vanishes.

Hausdorff dimension (D) is a non-negative real number to define the dimension of subsets of metric space.

If plane figure, embedded in three-dimensional space is covered by disks, then it can be covered by balls equator which are those discs.

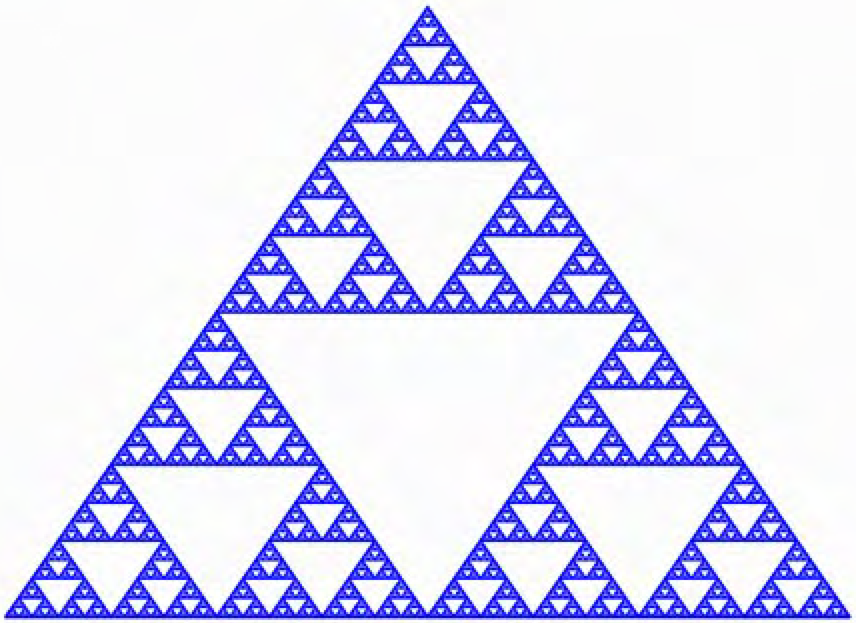
The area of figure $S = \sum s_i$, $s_i = \pi\rho^2$ where ρ is a radius of disk (or equator of ball if consider in 3-dimensional space). Here D = 2.

smooth curve D = 1, smooth surface D = 2, set of non-zero 3-volume D = 3.

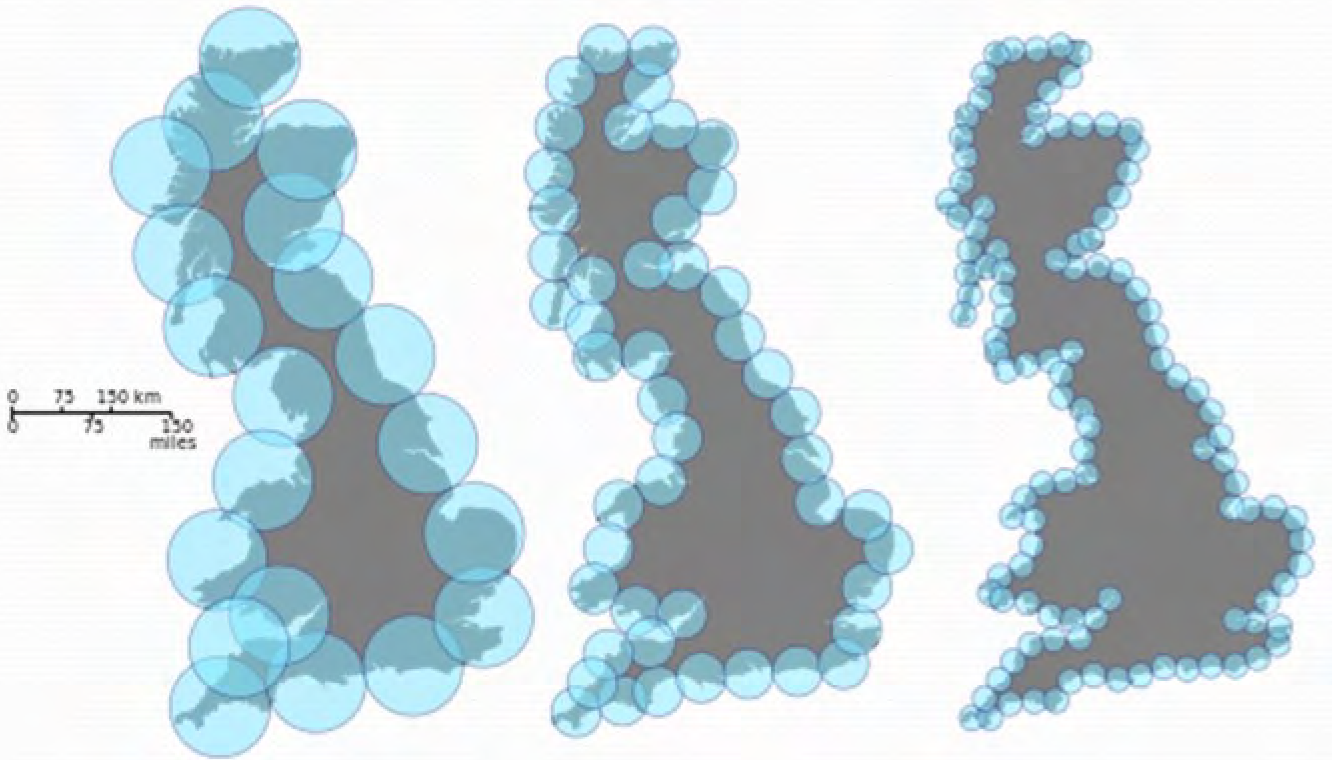


Julia set.

Examples of fractals



Sierpinski carpet, a set of fractal dimension 1.5849625.



Hausdorff dimension of the coast of Great Britain D = 1.25.

Box-counting algorithm

The box-counting dimension *Do* (fractal dimension or Hausdorff dimension) of a point distribution

$$D_0 = \lim_{r \to 0} \frac{\log N(r)}{\log 1/r}$$

where *N(r)* is the number of cubes of side *r*, needed to cover the set.

Hausdorff dimension does not exceed the *Do*, and for self-similar fractals they coincide.

Kolmogorov capacity vs correlation function

A ball of radius ε contains an object in compact subset M of R^3. Probability of finding an object in volume element

$$dP = \rho(r) dr d\Omega$$

$$V = 4\pi \int_M \rho(r) dr$$

$$N(\varepsilon) = V / \int_0^\varepsilon \rho(r) dr$$

$N(\varepsilon)$ is number of balls of radius ε in M.

Kolmogorov capacity of M

$$\dim_K M = \lim_{\varepsilon\to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\ln \int_0^\varepsilon \rho(r)dr}{\ln \varepsilon} = \lim_{\varepsilon\to 0} \frac{\varepsilon\rho(\varepsilon)}{\int_0^\varepsilon \rho(r)dr} = 1+\lim_{\varepsilon\to 0} \frac{\varepsilon\rho'(\varepsilon)}{\rho(\varepsilon)}$$

For power law density

$$\rho(r) \propto r^\alpha$$

the Kolmogorov capacity

$$\dim_K M = 1 + \alpha$$

When

$$\rho(r) \propto r^{\alpha} + r^{\beta}$$

$$\frac{\varepsilon \rho'(\varepsilon)}{\rho(\varepsilon)} = \frac{\alpha \varepsilon^{\alpha} + \beta \varepsilon^{\beta}}{\varepsilon^{\alpha} + \varepsilon^{\beta}}$$

Kolmogorov capacity depends on smaller index

$$\dim_K M = 1 + \min\{\alpha, \beta\}$$

Probability

$$dP = (1+\xi) r^2 dr d\Omega$$

at $\xi \propto r^{-\gamma}$ the density is

$$\rho = r^2 + \xi r^2 \propto r^2 + r^{2-\gamma}$$

Then

$$\dim_K M = 3 - \gamma$$

The Minimal Spanning Tree

Construction of a tree

Tree with no loops and with minimum total edge length.

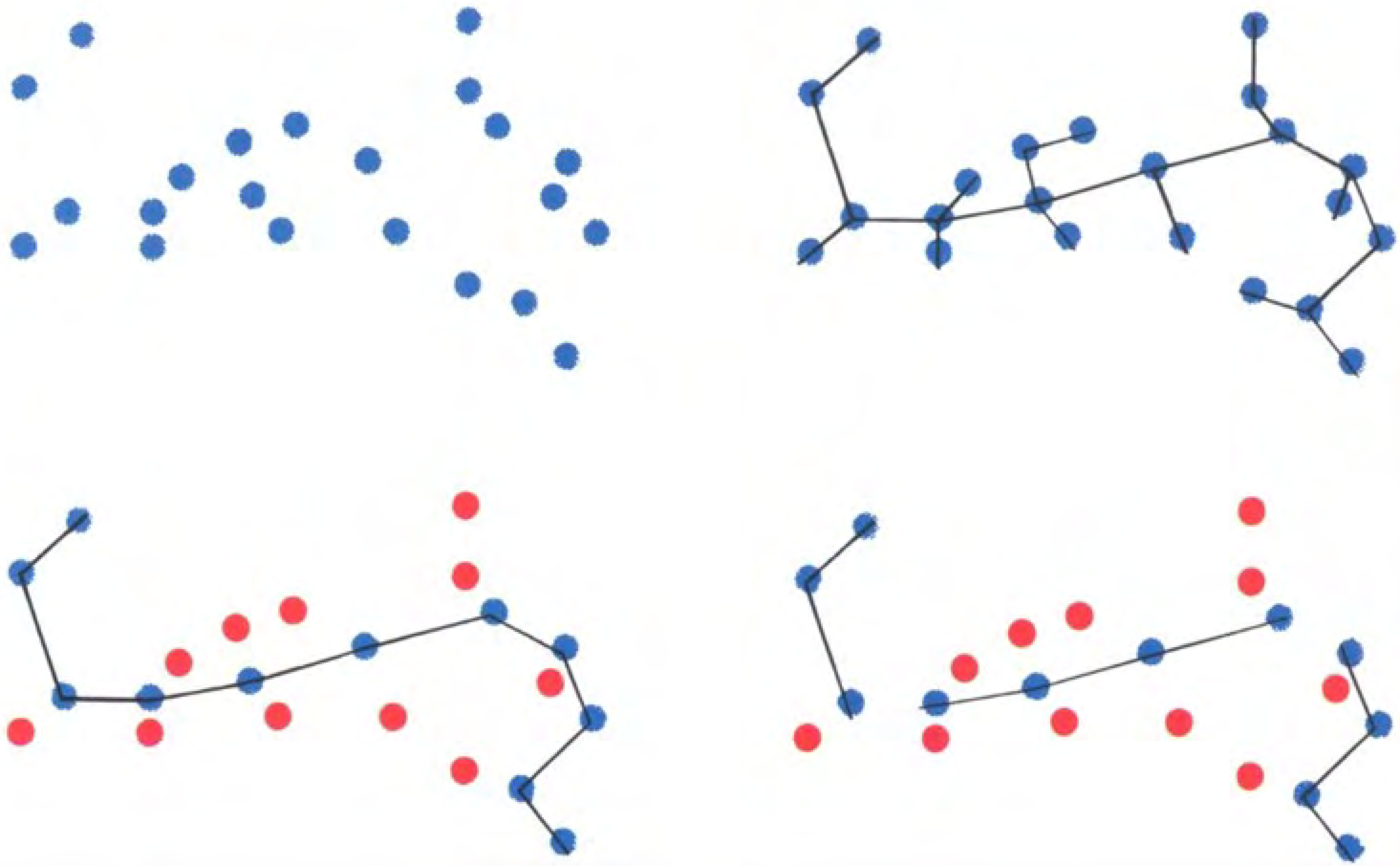
Pruning

Leave only those nodes which belong to the branch.

Separation

All edges of length exceeding some critical value are removed.

Numerical studies indicate no evidence for branches in case of Poisson distribution of points.



An example of Minimal spanning tree.

Another approach for Minimal Spanning Tree

Consider the lengths of $m = n - 1$ branches of tree $\{\ell_i\}_{i=1}^{m}$ as diameters of covering . If the moments of the lengths behave as

$$\sum_{i=1}^{m} \ell_i^{\beta}(m) \sim m^{1-\beta/h(\beta)}$$

when number of random points n changes, $h(\beta)$ defines the Hausdorff dimension.

Multi-fractals and Renyi dimension

Sets which need more than one single dimension to be fully characterized are multi-fractals.

Multi-fractals can be interpreted as mixture of simple fractals, each one characterized by its own fractal dimension and the whole set described by an infinite family of exponents.

Renyi or generalized dimensions

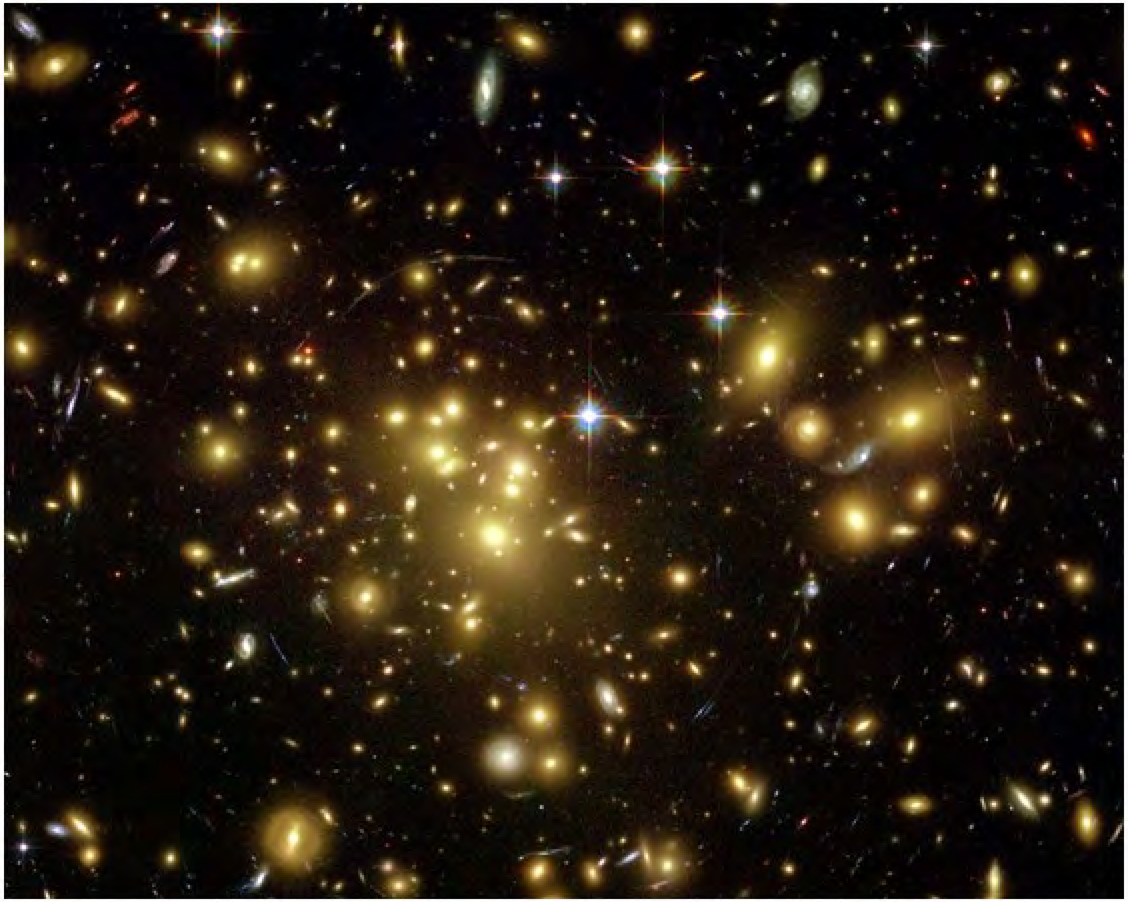
$$D_q = \lim_{r \to 0} (q-1)^{-1} \frac{\log \sum_{i=1}^{N(r)} \rho_i(r)^q}{\log r} \quad \text{if} \quad q \neq 1$$

$$D_1 = \lim_{r \to 0} \frac{\sum_{i=1}^{N(r)} \rho_i(r) \log \rho_i(r)}{\log r} \quad \text{if} \quad q = 1$$

$\rho_i(r)$ is probability that a point is in cell, and the set has been covered by disjoint cells of equal size r.

For homogeneous fractal $D = D_o$ for all q, instead in a multi-fractals, it is a monotonic decreasing function of q.

Fractal Dimension of galaxy distribution



Cluster of galaxies.

Correlation integral

Correlation integral *C(r)* is given by

$$C(r) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \Theta(r - |X_i - X_j|)$$

$$\theta(x) = \begin{cases} 0, & x < 0; \\ \frac{1}{2}, & x = 0; \\ 1, & x > 0. \end{cases}$$

θ is Heaveside step function and N is the total number of points in the distribution. *C(r)* measures the probability that two randomly selected points X_i and X_j of positions are closer than distance r.

Hausdorff dimension and two-point correlation function

For fractal distribution

$$\lim_{r \to 0} C(r) = r^{D_2}$$

D_2 is the Hausdorff dimension of the set.

Simple fractals $D_2 = D_0$, multi-fractals $D_2 < D_0$

D_2 provides a lower limit to the Hausdorff dimension.

$$N\delta P = n\left(1 + \xi(r)\right)\delta V$$

Relationship between integral correlation *C(r)* and two-point correlation functions $\xi(r)$:

$$1 + \xi(r) = \frac{N}{4\pi n r^2}\frac{dC(r)}{dr}$$

Other formula to compute $\xi(r)$

$$1 + \xi(r) = \frac{\Delta N(r)}{\Delta N_R(r)}$$

$\Delta N_R(r)$ is number of galaxies in ΔV ,

$\Delta N(r)$ is the number of galaxies in ΔV at distance *r*.

$$1+\xi(r)=\frac{dC(r)/dr}{dC_R(r)/dr}$$

where $C_R(r)$ is correlation integral for random distribution.

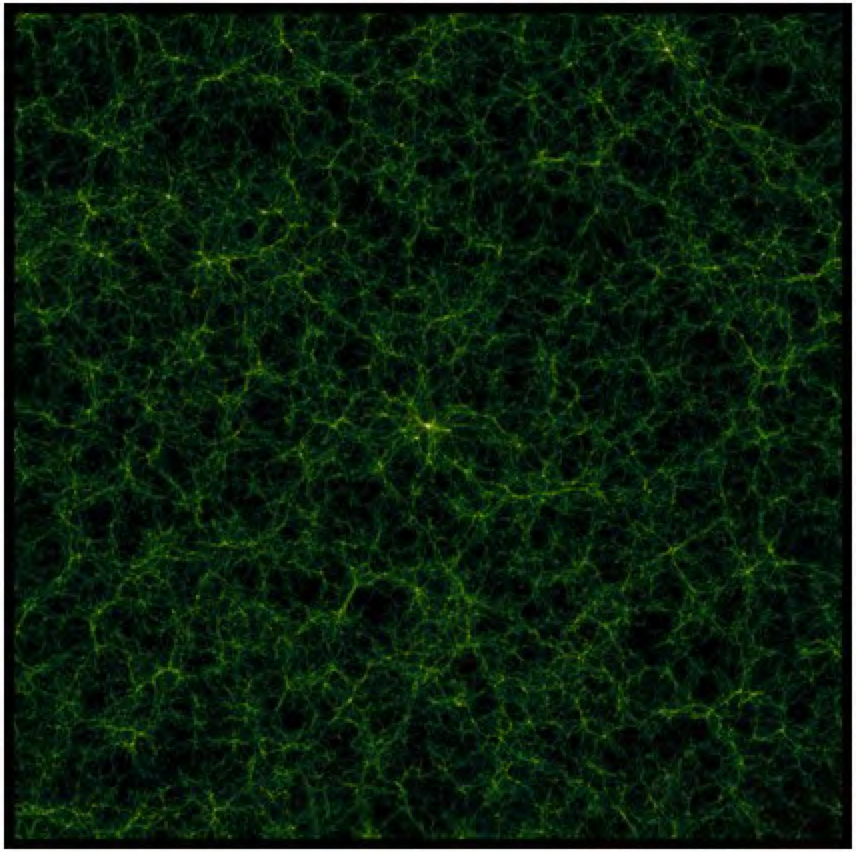
For small r reduced two-point correlation function's observational result is

$$\xi(r)\approx(r/r_0)^{-\gamma}$$

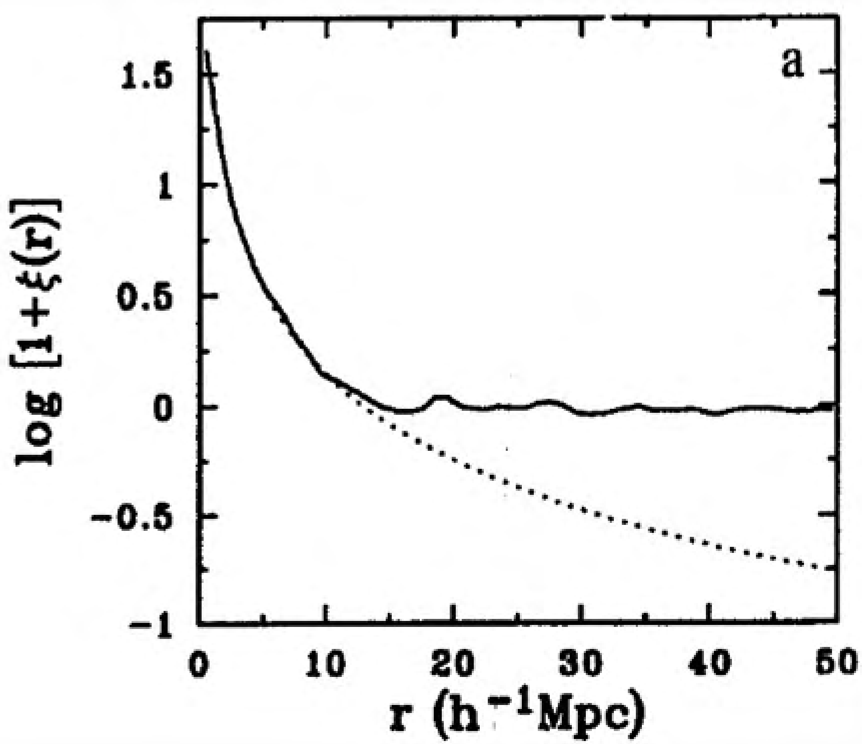
$$\gamma=1.77\pm0.04\,,\qquad r_0=5.4\pm1\,h^{-1}\,\mathrm{Mpc}$$

From formula for $\xi(r)$ we have $\xi(r)\approx r^{D_2-3}$

and the galaxy distribution on small scales has fractal dimension $D_2 = 1.23 \pm 0.04$.



This image of the large-scale universe is a slice from a large simulation called 'GiggleZ' which complements the WiggleZ survey. It shows a snapshot of the large-scale matter distribution. Released Aug. 21, 2012



The correlation function for the pancake model (solid line). The dotted line shows fractal behavior.

Hausdorff dimension, KS-entropy and Lyapunov exponents

Consider the set M(p) of points in interval [0,1] that contain 1 in their diadic expansion with probability $0 \leq p \leq 1$:

$$\dim_H M(p_0, \ldots, p_{r-1}) = \left(-\sum_{i=0}^{r-1} p_i \ln p_i \right) / \ln r \equiv \frac{h}{\ln r}$$

Where r is the possible states and p_i the corresponding probability of h is the KS-entropy, $\dim_H M$ is the Hausdorff dimension (H-dim).
This can be used to calculate the H-dim of Cantor set, for example.
H-dim of M reflects the degree of uncertainty in choosing the point from M.

Cantor set

$$\dim_H M = \frac{\ln 2}{\ln 3}$$

Kaplan-Yorke conjecture

for dissipative systems we have Lyapunov dimension:

$$\dim_\Lambda M = k + \frac{\lambda_1 + \ldots + \lambda_k}{|\lambda_{k+1}|}$$

where

$$k = \max\{i \mid \lambda_1 + \ldots + \lambda_i > 0\}$$

$$\lambda_1 > \lambda_2 > \ldots > \lambda_{max}$$

for arbitrary strange attractors

$$\dim_H M = \dim_\Lambda M$$

Self –similar β model

β model is three dimensional distribution of points generated by a sequence of breaking iterations starting from a parent object (e.g. a cube) into smaller objects:

$$l_1 = l_0 / M^{1/3} \qquad < m > = pM$$

l_0 is the linear size of parent cube, l_1 is the linear size of son cube, M is the count of son cubes, $< m >$ is the mean counts of active objects and p is the "survival probability", k denotes the k-th iteration.

Fractal dimension of the distribution is depend on "survival probability"

$$D_0 = \lim_{k \to \infty} \frac{\log \ < m >^k}{\log \ 2^k} = \frac{\log \ < m >}{\log \ 2} = \frac{\log \ pM}{\log \ 2}$$

Conclusions

Correlation functions are efficient tools for statistical description of various N-body systems, including of galaxy clusters, the key blocks of the large-scale Universe.

Fractals are unique structures appearing in various physical problems, including in the description of the large scale galaxy distribution, interstellar matter, etc.

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